

Nonfillable Legendrian knots in the 3-sphere

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1 Introduction

Let Λ be a Legendrian knot in the standard contact 3-sphere. If Λ bounds an exact orientable Lagrangian surface Σ in the standard symplectic 4-ball, then the genus of Σ is equal to the slice genus of (the smooth knot underlying) Λ , the rotation number of Λ is zero as well as the sum of the Thurston-Bennequin number of Λ and the Euler characteristic of Σ by a theorem of Chantraine [2]. Moreover the linearized contact homology of Λ with respect to the augmentation induced by Σ is isomorphic to the (singular) homology of Σ by a theorem of Seidel [7, 6]. Question (8.9) in [8] asks whether every augmentation for which Seidel's isomorphism holds is induced by a Lagrangian filling. We give a negative answer to this question based on the family of Legendrian knots

$$\{\Lambda_{p,q,r,s} : p, q, r, s \geq 2, p \equiv q \equiv r + 1 \equiv s + 1 \pmod{2}\}$$

given by the Lagrangian projection in Fig. (1). Throughout the paper we will always be working under the above assumption that the parities of p and q match and they are the opposite of those of r and s . The rotation and Thurston-Bennequin numbers of $\Lambda_{p,q,r,s}$ are 0 and 5, respectively. This gives a lower bound of 3 on the slice genus. On the other hand, the Seifert surface we obtain from this projection has genus 3 hence both the slice genus and the Seifert genus are 3.

Theorem 1 *The Chekanov-Eliashberg dg-algebra of $\Lambda_{p,p,p+1,p+1}$ admits an augmentation which is not induced by any exact orientable Lagrangian filling, although the corresponding linearized contact homology is isomorphic to the homology of a surface of genus 3 with one boundary component.*

Our examples are inspired by a deformation argument in [10] related to the Chekanov-Eliashberg algebra where the Legendrian link of unknots linked according to the D_4 tree is shown to be significant, specifically over a base field of characteristic 2 (see the last part of the proof of Thm.(14) in [10] and Rem. (7) below). As can be seen in Fig. (1), $\Lambda_{p,q,r,s}$ is constructed from the D_4 link by adding twists that turn it into a Legendrian knot. A similar construction was previously used in [4] on a different link to produce infinite families of Legendrian knots not isotopic to their Legendrian mirrors.

In the next section we describe the Chekanov-Eliashberg algebra and compute the linearized contact homology of our examples. In Sec. (3) we gather enough information on the strictly unital A_∞ -algebras obtained by dualizing the Chekanov-Eliashberg algebras of our Legendrian knots to prove that they are not quasi-isomorphic to the A_∞ -algebra of cochains on a closed surface whenever the base field has characteristic 2. This proves the nonfillability of our examples by a duality result of Ekholm and Lekili from their recent preprint [9] (see Thm. (3) below).

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2 Linearized contact homology of $\Lambda_{p,q,r,s}$

The Chekanov-Eliashberg algebra of a Legendrian knot is a differential graded algebra generated by Reeb chords from the knot to itself. We refer to [3] for the combinatorial description of the Chekanov-Eliashberg algebra of a Legendrian knot in the standard contact 3-sphere.

We denote the Chekanov-Eliashberg algebra of $\Lambda = \Lambda_{p,q,r,s}$ over a field \mathbb{K} by (L, ∂) . Since the rotation number of Λ is 0, we have a \mathbb{Z} -grading on (L, ∂) .

The generators of (L, ∂) are as indicated in Fig. (1):

$$\begin{aligned} & a_0^x, \dots, a_p^x, a_0^y, \dots, a_q^y, a_0^z, \dots, a_r^z, a_0^w, \dots, a_s^w \\ & x_0, \dots, x_p, y_0, \dots, y_q, z_0, \dots, z_r, w_0, \dots, w_s \\ & a_0, b_1, \dots, b_6 \end{aligned}$$

with gradings

$$\begin{aligned} & |*_i| = 0, |a_0| = |a_i^*| = 1, \text{ for } * \in \{x, y, z, w\} \\ & |b_1| = -|b_4| = p - r + 1, |b_2| = -|b_5| = q - r + 1, |b_3| = -|b_6| = r - s \end{aligned}$$

Let $\epsilon : L \rightarrow \mathbb{K}$ be the augmentation which maps all $*_i$ to -1 for $* \in \{x, y, z, w\}$.

Proposition 2 *There is an isomorphism*

$$HC_*^\epsilon(\Lambda_{p,p,p+1,p+1}) \cong H_{1-*}(\Sigma; \mathbb{K})$$

between the linearized contact homology of $\Lambda_{p,p,p+1,p+1}$ with respect to the augmentation ϵ and the homology of the orientable surface Σ of genus 3 with one boundary component.

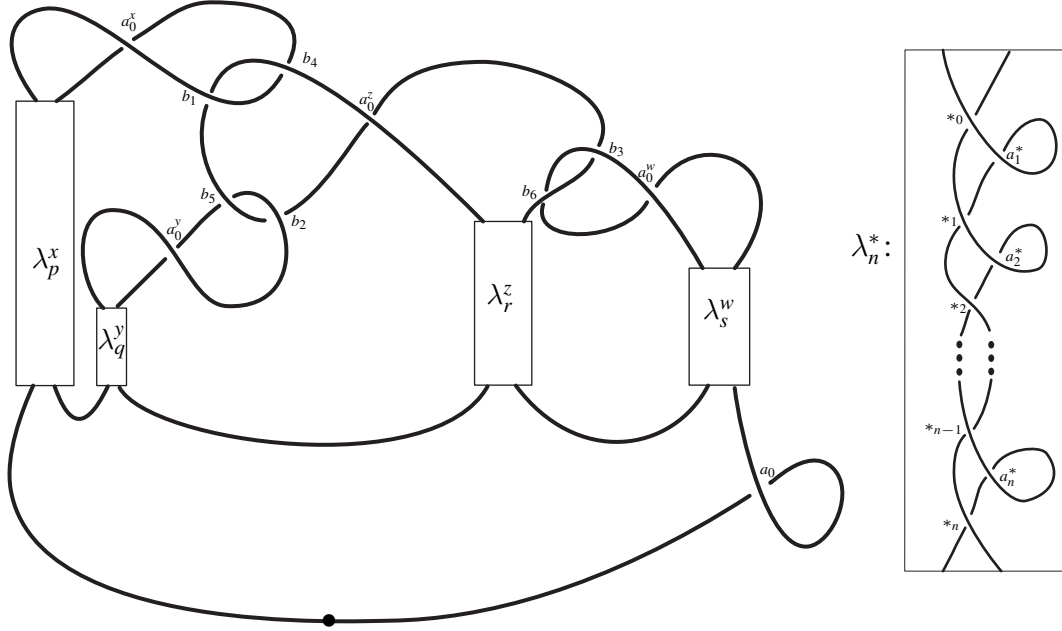


Figure 1: Lagrangian projection of $\Lambda_{p,q,r,s}$ for $p, q, r, s \geq 2$ and $p \equiv q \equiv r + 1 \equiv s + 1 \pmod{2}$

Proof Counting the relevant immersed polygons with the choice of a base-point on $\Lambda_{p,q,r,s}$ as indicated by \bullet in Fig. (1) we see that nontrivial differentials are given by

$$\begin{aligned}
 \partial a_0 &= 1 - w_s z_r y_q x_p \\
 \partial a_0^x &= 1 + x_0 + b_1 b_4 \\
 \partial a_0^y &= 1 + y_0 + b_2 b_5 \\
 \partial a_0^z &= 1 + z_0 + b_4 b_1 + b_5 b_2 + z_0 b_6 b_3 + b_4 b_1 b_5 b_2 \\
 \partial a_0^w &= 1 + w_0 + b_3 b_6 \\
 \partial a_i^* &= 1 - *_{i-1} *_i \text{ for } * \in \{x, y, z, w\} \text{ and } i \geq 1
 \end{aligned}$$

Conjugating ∂ by the automorphism $id + \epsilon$ gives another differential ∂^ϵ on L

$$\begin{aligned}
 \partial^\epsilon a_0 &= w_s + z_r + y_q + x_p - w_s z_r - w_s y_q - w_s x_p - z_r y_q - z_s x_p - y_q x_p \\
 &\quad + w_s z_r y_q + w_s z_r x_p + w_s y_q x_p + z_r y_q x_p - w_s z_r y_q x_p \\
 \partial^\epsilon a_0^x &= x_0 + b_1 b_4, \quad \partial^\epsilon a_0^y = y_0 + b_2 b_5, \quad \partial^\epsilon a_0^w = w_0 + b_3 b_6 \\
 \partial^\epsilon a_0^z &= z_0 + b_4 b_1 + b_5 b_2 - b_6 b_3 + z_0 b_6 b_3 + b_4 b_1 b_5 b_2 \\
 \partial^\epsilon a_i^* &= *_{i-1} + *_{i-1} *_i - *_{i-1} *_i, \text{ for } * \in \{x, y, z, w\} \text{ and } i \geq 1
 \end{aligned}$$

Applying the elementary transformation

$$a_0 \mapsto a_0 - (-1)^p \left(\sum_{i=0}^p (-1)^i a_i^x + \sum_{i=0}^q (-1)^i a_i^y - \sum_{i=0}^r (-1)^i a_i^z - \sum_{i=0}^s (-1)^i a_i^w \right)$$

simplifies the computation of linearized contact homology $HC_*^\epsilon(\Lambda)$ of Λ associated to the augmentation ϵ and more importantly, the description of the A_∞ -algebras that will be discussed in the next section.

At this point, we have the following presentation of the differential ∂_1^ϵ on the linearized complex which computes $HC_*^\epsilon(\Lambda_{p,q,r,s})$:

$$\begin{aligned} \partial_1^\epsilon a_0 &= \partial_1^\epsilon b_j = 0 \\ \partial_1^\epsilon a_0^x &= x_0, \quad \partial_1^\epsilon a_0^y = y_0, \quad \partial_1^\epsilon a_0^z = z_0, \quad \partial_1^\epsilon a_0^w = w_0 \\ \partial_1^\epsilon a_i^* &= *_{i-1} + *_{i-1}, \quad \text{for } * \in \{x, y, z, w\} \text{ and } i \geq 1 \end{aligned}$$

It is clear that $HC_*^\epsilon(\Lambda_{p,q,r,s})$ is spanned by a_0, b_1, \dots, b_6 . Moreover, if $p = q = r - 1 = s - 1$, then $|b_i| = 0$ for all i and we get the graded isomorphism in the statement. \square

3 The augmentation ϵ is not induced by a Lagrangian filling

In this section, we prove that $\Lambda_{p,q,r,s}$ has no exact Lagrangian filling associated to the augmentation ϵ by using a result from a recent preprint of Ekholm and Lekili.

The following is a consequence of [9, Thm.(4)].

Theorem 3 (Ekholm-Lekili) *If Λ has an exact Lagrangian filling Σ , then there is an A_∞ quasi-isomorphism between $\mathrm{RHom}_{CE^*}(\mathbb{K}, \mathbb{K})$ and the A_∞ -algebra $C^*(S; \mathbb{K})$ of (singular) cochains on the closed surface S obtained by capping the boundary of Σ , where $CE^* = L_{-*}$ and \mathbb{K} is equipped with a CE -module structure by the augmentation $\epsilon_\Sigma : CE \rightarrow \mathbb{K}$ induced by the filling Σ .*

In order to describe the A_∞ -algebra $\mathrm{RHom}_{CE^*}(\mathbb{K}, \mathbb{K})$ in the above statement, we utilize the isomorphism between $\mathrm{RHom}_{CE^*}(\mathbb{K}, \mathbb{K})$ and the linear dual \mathcal{B} of the Legendrian A_∞ -coalgebra defined in the more general setting of [9]. In the current setup, the strictly unital A_∞ -algebra \mathcal{B} can be obtained from the non-unital A_∞ -algebra on the linearized cochain complex defined in [4] (which is also the endomorphism algebra of ϵ in the Aug_- category of [1]) by adding a copy of \mathbb{K} to make it unital (cf. [9, Rem.(24)]).

The description of $\mathcal{B} \cong \mathrm{RHom}_{CE^*}(\mathbb{K}, \mathbb{K})$ we provide is based on the presentation of (L, ∂^ϵ) obtained at the end of the proof of Prop. (2). We abuse the notation and denote the duals of the generators of L

by the generators themselves. The nontrivial A_∞ -products on \mathcal{B} (besides those dictated by strict unitality) are

$$\begin{aligned}
\mu_{\mathcal{B}}^1(*_i) &= a_i^* + a_{i+1}^*, \text{ for } * \in \{x, y, z, w\} \text{ and } i \geq 0 \\
\mu_{\mathcal{B}}^1(x_p) &= a_p^x, \mu_{\mathcal{B}}^1(y_q) = a_q^y, \mu_{\mathcal{B}}^1(z_r) = a_r^z, \mu_{\mathcal{B}}^1(w_s) = a_s^w \\
\mu_{\mathcal{B}}^2(b_1, b_4) &= (-1)^{p+1}a_0 + a_0^x, \mu_{\mathcal{B}}^2(b_2, b_5) = (-1)^{p+1}a_0 + a_0^y, \mu_{\mathcal{B}}^2(b_3, b_6) = (-1)^p a_0 + a_0^w \\
\mu_{\mathcal{B}}^2(b_4, b_1) &= \mu_{\mathcal{B}}^2(b_5, b_2) = (-1)^p a_0 + a_0^z, \mu_{\mathcal{B}}^2(b_6, b_3) = (-1)^{p+1}a_0 - a_0^z \\
\mu_{\mathcal{B}}^2(x_{i-1}, x_i) &= (-1)^{p+i}a_0 - a_i^x, \mu_{\mathcal{B}}^2(y_{i-1}, y_i) = (-1)^{p+i}a_0 - a_i^y \\
\mu_{\mathcal{B}}^2(z_{i-1}, z_i) &= (-1)^{p+i+1}a_0 - a_i^z, \mu_{\mathcal{B}}^2(w_{i-1}, w_i) = (-1)^{p+i+1}a_0 - a_i^w \\
\mu_{\mathcal{B}}^2(w_s, z_r) &= \mu_{\mathcal{B}}^2(w_s, y_q) = \mu_{\mathcal{B}}^2(w_s, x_p) = \mu_{\mathcal{B}}^2(z_r, y_q) = \mu_{\mathcal{B}}^2(z_r, x_p) = \mu_{\mathcal{B}}^2(y_q, x_p) = -a_0 \\
\mu_{\mathcal{B}}^3(z_0, b_6, b_3) &= a_0 + a_0^z \\
\mu_{\mathcal{B}}^3(w_s, z_r, y_q) &= \mu_{\mathcal{B}}^3(w_s, z_r, x_p) = \mu_{\mathcal{B}}^3(w_s, y_q, x_p) = \mu_{\mathcal{B}}^3(z_r, y_q, x_p) = a_0 \\
\mu_{\mathcal{B}}^4(b_4, b_1, b_5, b_2) &= a_0 + a_0^z, \mu_{\mathcal{B}}^4(w_s, z_r, y_q, x_p) = -a_0
\end{aligned}$$

and the gradings in \mathcal{B} are

$$\begin{aligned}
|a_0| &= |a_i^*| = 2, \quad |*_i| = 1 \text{ for } * \in \{x, y, z, w\} \text{ and } i \geq 0 \\
|b_1| &= -|b_4| = p - r + 2, \quad |b_2| = -|b_5| = q - r + 2, \quad |b_3| = -|b_6| = r - s + 1
\end{aligned}$$

A homological perturbation argument, as suggested by Prop. (1.12) and Rem. (1.13) in [11], provides a minimal model $(\mathcal{A}, \mu_{\mathcal{A}}^\bullet)$ quasi-isomorphic to $(\mathcal{B}, \mu_{\mathcal{B}}^\bullet)$. From the above description of the A_∞ -products we see the decomposition $\mathcal{B} = \mathcal{A} \oplus \mathcal{C}$, where \mathcal{A} is generated by

$$\{1, a_0, b_i : i = 1, \dots, 6\}$$

and gives a minimal model for \mathcal{B} , whereas \mathcal{C} is the subalgebra generated by the rest of the generators of \mathcal{B} and acyclic with respect to the differential μ^1 . We choose a contracting homotopy $T^1 : \mathcal{C} \rightarrow \mathcal{C}$ with $\mu_{\mathcal{B}}^1 T^1 + T^1 \mu_{\mathcal{B}}^1 = F^1 G^1 - id$ as follows

$$\begin{aligned}
T^1(a_i^x) &= -x_i + x_{i+1} + \dots + (-1)^{p-i+1}x_p, \quad T^1(a_i^y) = -y_i + y_{i+1} + \dots + (-1)^{q-i+1}y_q \\
T^1(a_i^z) &= -z_i + z_{i+1} + \dots + (-1)^{r-i+1}z_r, \quad T^1(a_i^w) = -w_i + w_{i+1} + \dots + (-1)^{s-i+1}w_s
\end{aligned}$$

This homotopy T^1 and Eqn. (1.18) in [11] suffices to compute the A_∞ -products $\mu_{\mathcal{A}}^\bullet$ on \mathcal{A} . To begin with, the only nontrivial $\mu_{\mathcal{A}}^2$ products (besides those dictated by strict unitality) are

$$\begin{aligned}
\mu_{\mathcal{A}}^2(b_1, b_4) &= \mu_{\mathcal{A}}^2(b_2, b_5) = \mu_{\mathcal{A}}^2(b_6, b_3) = (-1)^{p+1}a_0 \\
\mu_{\mathcal{A}}^2(b_4, b_1) &= \mu_{\mathcal{A}}^2(b_5, b_2) = \mu_{\mathcal{A}}^2(b_3, b_6) = (-1)^p a_0
\end{aligned}$$

Moreover, $\mu_{\mathcal{A}}^3$ vanishes since

- $\mu_{\mathcal{B}}^3$ is trivial on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, and
- $\mu_{\mathcal{B}}^2$ vanishes on $\text{im}(F^2) \otimes \mathcal{A} \subset \text{im}(T^1) \otimes \mathcal{A} \subset \mathcal{C} \otimes \mathcal{A}$ and on $\mathcal{A} \otimes \text{im}(F^2) \subset \mathcal{A} \otimes \mathcal{C}$

where $F^2 = T^1 \circ \mu_{\mathcal{B}}^2$.

Proceeding further, Eqn. (1.18) in [11] for $d = 4$ simplifies as

$$\mu_{\mathcal{A}}^4(\alpha_4, \dots, \alpha_1) = G^1 \left(\mu_{\mathcal{B}}^4(\alpha_4, \dots, \alpha_1) + \mu_{\mathcal{B}}^3(F^2(\alpha_4, \alpha_3), \alpha_2, \alpha_1) + \mu_{\mathcal{B}}^2(F^2(\alpha_4, \alpha_3), F^2(\alpha_2, \alpha_1)) \right)$$

where $G^1 : \mathcal{B} \rightarrow \mathcal{A}$ is the projection.

At this point, one immediately sees that all three summands on the right hand side of the above formula for $\mu_{\mathcal{A}}^4$ vanish on quadruples which are not of the form $(b_i, b_{i\pm 3}, b_j, b_{j\pm 3})$. In order to prove the key proposition below, it suffices to compute $\mu_{\mathcal{A}}^4(b_i, b_j, b_k, b_l)$, where (b_i, b_j, b_k, b_l) is a cyclic permutation of (b_1, b_2, b_5, b_4) , (b_1, b_5, b_2, b_4) , or (b_2, b_5, b_2, b_5) . To this end, it is straightforward, if tedious, to check that, depending on the parity of p , the following are the only nontrivial ones among these ten μ^4 products:

$$\mu_{\mathcal{A}}^4(b_4, b_1, b_2, b_5) = a_0, \mu_{\mathcal{A}}^4(b_5, b_2, b_4, b_1) = \mu_{\mathcal{A}}^4(b_5, b_2, b_5, b_2) = -a_0$$

if p is even, and

$$\mu_{\mathcal{A}}^4(b_4, b_1, b_2, b_5) = \mu_{\mathcal{A}}^4(b_4, b_1, b_5, b_2) = a_0, \mu_{\mathcal{A}}^4(b_2, b_5, b_2, b_5) = -a_0$$

if p is odd.

Besides the computations above, another important ingredient for the proof of the proposition below is the following formality statement.

Lemma 4 *Over a field \mathbb{K} of characteristic 2, the algebra of cochains $C^*(S; \mathbb{K})$ on a closed orientable surface S is a formal differential graded algebra.*

Over the base field $\mathbb{K} = \mathbb{R}$, there is the classical (and much more general) formality result in [5]. Since we were not able to locate an extension of this result to nonzero characteristic cases in the literature, we provide a proof of Lem. (4) at the end of this section. In fact, the characteristic condition in the statement of Lem. (4) can be removed by a straightforward modification of the proof.

Proposition 5 *If the characteristic of the base field \mathbb{K} is 2, then the A_{∞} -algebra \mathcal{A} is not A_{∞} quasi-isomorphic to the algebra of cochains on a closed orientable surface S .*

Proof By Lem. (4), it suffices to prove that there is no A_{∞} quasi-isomorphism between \mathcal{A} and $H^*(S; \mathbb{K})$. Suppose that \mathcal{F} is an A_{∞} -algebra homomorphism from \mathcal{A} to $H^*(S; \mathbb{K})$. Since \mathcal{A} is a

minimal A_∞ -algebra, $\mu_{\mathcal{A}}^1$ is trivial. We have also established above that $\mu_{\mathcal{A}}^3$ vanishes as well. As a consequence, a particular set of A_∞ -functor equations, satisfied by the family of graded multilinear maps $\mathcal{F}^d : \mathcal{A}^{\otimes d} \rightarrow H^*(S; \mathbb{K})[1-d]$, simplifies to

$$\begin{aligned} \mathcal{F}^1(\mu_{\mathcal{A}}^4(b_i, b_j, b_k, b_l)) &= \mathcal{F}^3(b_i, b_j, b_k) \cup \mathcal{F}^1(b_l) + \mathcal{F}^1(b_i) \cup \mathcal{F}^3(b_j, b_k, b_l) + \mathcal{F}^2(b_i, b_j) \cup \mathcal{F}^2(b_k, b_l) \\ &\quad + \mathcal{F}^3(\mu_{\mathcal{A}}^2(b_i, b_j), b_k, b_l) + \mathcal{F}^3(b_i, \mu_{\mathcal{A}}^2(b_j, b_k), b_l) + \mathcal{F}^3(b_i, b_j, \mu_{\mathcal{A}}^2(b_k, b_l)) \end{aligned}$$

In the rest of the proof we refer to the above equation as $Eq_{(i,j,k,l)}$ and consider the sum of all the equations $Eq_{(i,j,k,l)}$ where (i,j,k,l) is a cyclic permutation of $(1, 2, 5, 4)$, $(1, 5, 2, 4)$ or $(2, 5, 2, 5)$.

First of all, the computation preceding this proposition implies that the sum of the left hand side of these ten equations is equal to $(-1)^{p+1} \mathcal{F}^1(a_0)$. In contrast, the right hand side of the sum of these ten equations is 0. Once we establish this, we get $\mathcal{F}^1(a_0) = 0$ and hence \mathcal{F} is not a quasi-isomorphism.

To prove the vanishing of the right hand side we consider the terms on the right hand side in three separate groups and argue that each group adds up to 0 under the assumption that $\text{char}(\mathbb{K}) = 2$. First observe that, since the cup product \cup is (graded-)commutative, each of the first two terms on the right hand side of the equation $Eq_{(i,j,k,l)}$ appears in exactly one other equation, namely $Eq_{(l,i,j,k)}$ or $Eq_{(j,k,l,i)}$. For the same reason, the third term on the right hand side of $Eq_{(i,j,k,l)}$ is cancelled by that of $Eq_{(k,l,i,j)}$, unless of course $(i,j) = (k,l)$. This leaves us with the sum of the third terms of $Eq_{(2,5,2,5)}$ and $Eq_{(5,2,5,2)}$,

$$\mathcal{F}^2(b_2, b_5) \cup \mathcal{F}^2(b_2, b_5) + \mathcal{F}^2(b_5, b_2) \cup \mathcal{F}^2(b_5, b_2)$$

which vanish by the general properties of the cup product. Finally, remember that we always have $\mu_{\mathcal{A}}^2(b_i, b_j) = 0$ for $|i-j| \neq 3$ and, if $\text{char}(\mathbb{K}) = 2$,

$$\mu_{\mathcal{A}}^2(b_4, b_1) = \mu_{\mathcal{A}}^2(b_2, b_5) = \mu_{\mathcal{A}}^2(b_5, b_2) = a_0.$$

This suffices to conclude that each of the last three terms on the right hand side of any one of the equations is either 0 or it appears in exactly two of our equations, e.g. the fifth term in $Eq_{(2,4,1,5)}$ is equal to the fifth term in $Eq_{(2,5,2,5)}$. \square

Corollary 6 *The Legendrian knot $\Lambda_{p,q,r,s}$ admits an augmentation which is not induced by an exact orientable Lagrangian filling.*

Remark 7 *When $\text{char}(\mathbb{K}) \neq 2$ our proof of Prop. (5) breaks down because the right hand side of the sum of the ten equations we consider in the last step of the proof is equal to*

$$2(-1)^p (\mathcal{F}^3(a_0, b_2, b_5) + \mathcal{F}^3(b_2, a_0, b_5) + \mathcal{F}^3(b_2, b_5, a_0))$$

which is not necessarily 0 in general.

Proof (of Lem. (4)) We prove the formality of the differential graded algebra $C = C^*(S, \mathbb{K})$ of (simplicial) cochains with the cup product on the closed surface S associated to the triangulation given in Fig. (2) by providing a zig-zag of explicit dg-algebra quasi-isomorphisms connecting C and the cohomology algebra $H = H^*(S, \mathbb{K})$ of S .

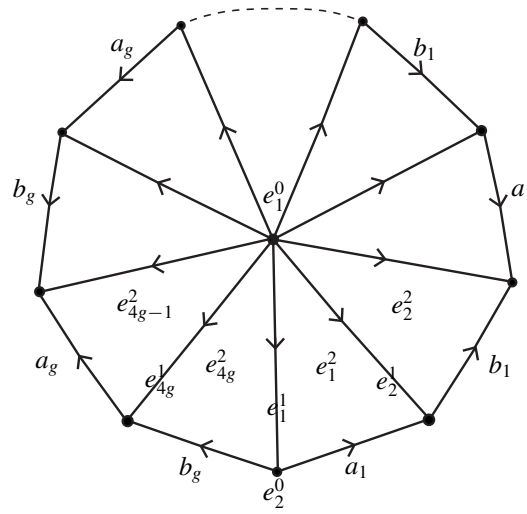


Figure 2: A triangulation of a closed, orientable surface S of genus g

We denote the generators of C by

$$e_i, \alpha_j, \beta_j, \theta_k, \gamma_k \quad \text{for } 1 \leq j \leq g, \quad 1 \leq k \leq 4g$$

which represent the duals of the simplices

$$e_i^0, a_j, b_j, e_k^1, e_k^2$$

as indicated in Fig. (2). The nontrivial differentials and products can be read from the triangulation as

$$\partial e_1 = \partial e_2 = \theta_1 + \cdots + \theta_{4g}$$

$$\partial \alpha_j = \gamma_{4j-3} + \gamma_{4j-1}, \quad \partial \beta_j = \gamma_{4j-2} + \gamma_{4j}, \quad \partial \theta_k = \gamma_{k-1} + \gamma_k$$

and

$$e_i e_i = e_i, \quad e_1 \theta_k = \theta_k, \quad e_1 \gamma_k = \gamma_k, \quad \theta_k e_2 = \theta_k, \quad \gamma_k e_2 = \gamma_k$$

$$e_2 \alpha_j = \alpha_j e_2 = \alpha_j, \quad e_2 \beta_j = \beta_j e_2 = \beta_j$$

$$\theta_{4j-3} \alpha_j = \gamma_{4j-3}, \quad \theta_{4j-2} \beta_j = \gamma_{4j-2}$$

$$\theta_{4j} \alpha_j = \gamma_{4j-1}, \quad \theta_{4j+1} \beta_j = \gamma_{4j}$$

(In the above equations and the rest of the proof, indices should always be interpreted modulo $4g$.)

We now define another dg-algebra, quasi-isomorphic to C and with a simplified differential so that the rest of the proof is more transparent. This new dg-algebra C' is generated by

$$e, \varphi_j, \psi_j, \nu, \epsilon_1, \zeta_1, \xi_l, \nu_l \text{ for } 1 \leq j \leq g, 1 \leq l \leq 4g-1$$

so that the map $\Phi : C' \rightarrow C$ defined by

$$\begin{aligned} \Phi : \quad e &\mapsto e_1 + e_2, \quad \epsilon_1 \mapsto e_1, \quad \zeta_1 \mapsto \theta_1 + \cdots + \theta_{4g} \\ \varphi_j &\mapsto \alpha_j + \theta_{4j-2} + \theta_{4j-1}, \quad \psi_j \mapsto \beta_j + \theta_{4j-1} + \theta_{4j}, \\ \xi_l &\mapsto \theta_l, \quad \nu_l \mapsto \gamma_{l-1} + \gamma_l, \quad \nu \mapsto \gamma_{4g} \end{aligned}$$

is a dg-algebra quasi-isomorphism. More precisely, on C' , the nontrivial differentials are

$$\partial' \epsilon_1 = \zeta_1, \quad \partial' \xi_l = \nu_l$$

e is the identity element, and the remaining products are

$$\begin{aligned} \varphi_j \psi_j &= \nu + \nu_1 + \cdots + \nu_{4j-2}, \quad \psi_j \varphi_j = \nu + \nu_1 + \cdots + \nu_{4j-1} \\ \xi_{4j-3} \varphi_j &= \nu + \nu_1 + \cdots + \nu_{4j-3}, \quad \xi_{4j-2} \psi_j = \nu + \nu_1 + \cdots + \nu_{4j-2} \\ \text{for } j < g, \quad \xi_{4j} \varphi_j &= \nu + \nu_1 + \cdots + \nu_{4j-1}, \quad \xi_{4j+1} \psi_j = \nu + \nu_1 + \cdots + \nu_{4j}, \quad \xi_1 \psi_g = \nu \\ \epsilon_1 \epsilon_1 &= \epsilon_1, \quad \epsilon_1 \xi_l = \xi_l, \quad \epsilon_1 \varphi_j = \xi_{4j-2} + \xi_{4j-1} \\ \text{for } j < g, \quad \epsilon_1 \psi_j &= \xi_{4j-1} + \xi_{4j}, \quad \epsilon_1 \psi_g = \zeta_1 + \xi_1 + \cdots + \xi_{4g-2}, \quad \epsilon_1 \nu = \nu \\ \epsilon_1 \zeta_1 &= \zeta_1, \quad \epsilon_1 \nu_l = \nu_l, \quad \zeta_1 \varphi_j = \nu_{4j-2} + \nu_{4j-1} \\ \text{for } j < g, \quad \zeta_1 \psi_j &= \nu_{4j-1} + \nu_{4j}, \quad \zeta_1 \psi_g = \nu_1 + \cdots + \nu_{4g-2} \end{aligned}$$

In the next step, we define yet another dg-algebra \widehat{C} by stabilizing C' , i.e. \widehat{C} contains C' as a subalgebra and the inclusion map is a dg-algebra quasi-isomorphism. Namely, we add the generators

$$\epsilon_k, \zeta_k \text{ for } 2 \leq k \leq 2g+1$$

with

$$|\epsilon_k| = 0, \quad |\zeta_k| = 1 \text{ and } \widehat{\partial} \epsilon_k = \zeta_k, \quad \widehat{\partial} \zeta_k = 0$$

to those of C' . and extend the algebra structure to \widehat{C} by adding the following nontrivial products

$$\begin{aligned} \epsilon_{2j} \psi_j &= \xi_1 + \cdots + \xi_{4j-2}, \quad \epsilon_{2j+1} \varphi_j = \xi_1 + \cdots + \xi_{4j-1} \\ \zeta_{2j} \psi_j &= \nu_1 + \cdots + \nu_{4j-2}, \quad \zeta_{2j+1} \varphi_j = \nu_1 + \cdots + \nu_{4j-1} \end{aligned}$$

for $j = 1, \dots, g$.

Finally, it is clear that the map $\widehat{\Phi} : H \rightarrow \widehat{C}$ defined on the cohomology algebra $H = H^*(S; \mathbb{K})$ by

$$\widehat{\Phi} : \quad e \mapsto e, \quad \overline{\varphi}_j \mapsto \varphi_j + \zeta_{2j}, \quad \overline{\psi}_j \mapsto \psi_j + \zeta_{2j+1}, \quad \nu \mapsto \nu$$

is a dg-algebra quasi-isomorphism proving the formality of $C = C^*(S; \mathbb{K})$. \square

References

- [1] F. Bourgeois, B. Chantraine, Bilinearized Legendrian contact homology and the augmentation category. *J. Symplectic Geom.* 12 (2014) 553–583.
- [2] B. Chantraine, On Lagrangian concordance of Legendrian knots. *Algebr. Geom. Topol.* 10 (2010) 63–85.
- [3] Y. Chekanov, Differential algebra of Legendrian links. *Invent. Math.* 150 (2002) 441–483.
- [4] G. Civan, P. Koprowski, J. Etnyre, J. Sabloff, A. Walker, Product structures for Legendrian contact homology. *Math. Proc. Cambridge Philos. Soc.* 150 (2011) 291–311.
- [5] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, Real homotopy theory of Kähler manifolds. *Invent. Math.* 29 (1975) 245–274.
- [6] G. Dimitroglou Rizell, Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbb{R}$ and applications. *Quantum Topol.* 7 (2016) 29–105.
- [7] T. Ekholm, Rational SFT, linearized Legendrian contact homology, and Lagrangian Floer cohomology. *Perspectives in analysis, geometry, and topology*, 109–145, *Progr. Math.*, 296, Birkhäuser/Springer, New York, 2012.
- [8] T. Ekholm, K. Honda, T. Kálmán, Legendrian knots and exact Lagrangian cobordisms. *J. Eur. Math. Soc.* 18 (2016) 2627–2689.
- [9] T. Ekholm, Y. Lekili, Duality between Lagrangian and Legendrian invariants. [arXiv:1701.01284](https://arxiv.org/abs/1701.01284)
- [10] T. Evgü, Y. Lekili, Koszul duality patterns in Floer theory. *Geom. Topol.* (to appear), [arXiv:1502.07922](https://arxiv.org/abs/1502.07922)
- [11] P. Seidel, Fukaya categories and Picard-Lefschetz theory, *Zurich Lectures in Advanced Mathematics*, European Mathematical Society (EMS), Zürich, 2008.

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